A SIMPLE SEPARABLE EXACT C*-ALGEBRA NOT ANTI-ISOMORPHIC TO ITSELF

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Abstract. We give an example of an exact, stably finite, simple. separable C*-algebra D which is not isomorphic to its opposite algebra. Moreover, D has the following additional properties. It is stably finite, approximately divisible, has real rank zero and stable rank one, has a unique tracial state, and the order on projections over D is determined by traces. It also absorbs the Jiang-Su algebra Z, and in fact absorbs the 3^{∞} UHF algebra. We can also explicitly compute the K-theory of D, namely $K_0(D) \cong \mathbb{Z}\left[\frac{1}{3}\right]$ with the standard order, and $K_1(D) = 0$, as well as the Cuntz semigroup of D, namely $W(D) \cong \mathbb{Z}\left[\frac{1}{3}\right]_{+} \sqcup (0, \infty)$.

The construction can be generalized to any odd prime p such that -1 is not a square mod p, giving a C*-algebra D which has all the properties mentioned above, except D absorbs the p^{∞} UHF algebra and one has p in place of 3 in the formulas for the K-theory and Cuntz semigroup.

1. Introduction

A C*-algebra A is said to be anti-isomorphic to itself if there exists an isomorphism between A and its opposite algebra A^{op}. The opposite algebra is by definition the C*-algebra whose underlying vector space structure, norm, and adjoint are the same as for A, while the product of x and y is equal to yx instead of xy.

In this paper we give an example of a simple separable exact C*-algebra which is not anti-isomorphic to itself. There are several examples in the literature of factors of type II_1 and type III which are not isomorphic to their opposite algebras. (See for example [7], [35], and [8].) A C*-algebra isomorphism of von Neumann algebras is necessarily a von Neumann algebra isomorphism, by Corollary 5.13 of [32]. These are therefore also examples of simple C*-algebras not isomorphic to their opposite algebras. However, none of these examples is separable or exact in the C*-algebra sense. (An infinite factor contains a copy of every separable C*-algebra, so is not exact as a C*-algebra. A factor of type II_1 contains a copy of the C*-algebra M of all bounded sequences in $\prod_{k=1}^{n} M_k$, and the proof of the proposition in Section 2.5 of [36] shows that M is not exact.)

Date: 21 January 2010.

2000 *Mathematics Subject Classification.* 46L35, 46L37, 46L40, 46L54.

N. Christopher Phillips was partially supported by NSF grant DMS-0701076, by the Fields Institute for Research in Mathematical Sciences, Toronto, Canada, and by an Elliott Distinguished Visitorship at the Fields Institute. Maria Grazia Viola was partly supported by the Fields Institute for Research in Mathematical Sciences, and by Queen's University, Kingston, Canada.

In a recent work [25], the first author proved the existence of a simple separable C*-algebra A which is not isomorphic to its opposite algebra. Moreover, A was shown to have real rank zero and $K_1(A) = 0$. The C*-algebra A was constructed as a subalgebra of Connes' example of a II_1 factor N not anti-isomorphic to itself (see [7]), by finding a series of intermediate C*-algebras $(B_n)_{n \geq 0}$ in N with some nice properties relative to the trace and the real rank, and taking $A = \overline{\bigcup_{n=0}^{\infty} B_n}$. However, the construction did not give a clear description or an explicit model for this example. Problem 4.5 of [25] asked for a more natural and explicit example of a simple separable C*-algebra not isomorphic to its opposite algebra, and Problem 4.7 of [25] asked for an exact example. We provide an example satisfying both conditions here.

We construct our example using a C* analog of a variation of Connes' example of a II_1 factor not anti-isomorphic to itself. This variation, denoted by M, was described in [35], where the second author gave a subfactor construction of such a factor. We let $\mathbb{Z}_n$ denote $\mathbb{Z}/n\mathbb{Z}$. Our C*-algebra is a crossed product of a C*-algebra C_0 by an action γ of $\mathbb{Z}_3$ on C_0 such that γ has the Rokhlin property. We show that the $C_0 \rtimes_{\gamma} \mathbb{Z}_3$ is an exact, stably finite, simple, separable, unital C*-algebra. Moreover, using our explicit construction, we show that $C_0 \rtimes_{\gamma} \mathbb{Z}_3$ is approximately divisible and has stable rank one and real rank zero, and that it absorbs the Jiang-Su algebra Z and the 3^{∞} UHF algebra. Also, the order on projections over $C_0 \rtimes_{\gamma} \mathbb{Z}_3$ is determined by traces, and the K-groups of $C_0 \rtimes_{\gamma} \mathbb{Z}_3$ are given by

$$K_0(C_0 \rtimes_{\gamma} \mathbb{Z}_3) \cong \mathbb{Z}\left[\tfrac{1}{3}\right] \quad \text{and} \quad K_1(C_0 \rtimes_{\gamma} \mathbb{Z}_3) = 0.$$

The C*-algebra C_0 and the action γ are described in Section 4. The algebra C_0 is the tensor product of a reduced free product A_0 and the UHF algebra $B_0 = \varinjlim_n (M_{3^n}, \varphi_n)$, where φ_n is the embedding $a \mapsto a \otimes 1$ of M_{3^n} in $M_{3^{n+1}} = M_{3^n} \otimes M_3$. The action γ is a perturbation by an inner automorphism of the tensor product of two automorphisms $\alpha \in \mathrm{Aut}(A_0)$ and $\beta \in \mathrm{Aut}(B_0)$, which are described in Lemmas 4.6 and 4.3. We show in Section 6 that γ has the Rokhlin property. The Rokhlin property allows us to transfer several properties of the C*-algebra C_0 to the crossed product $D = C_0 \rtimes_{\gamma} \mathbb{Z}_3$. In particular, when the finite group action has the Rokhlin property, approximate divisibility, stable rank one, and real rank zero are preserved under the operation of taking crossed products. Moreover, we also use the Rokhlin property to prove that the C*-algebra D tensorially absorbs the 3^{∞} UHF algebra B_0 and the Jiang-Su algebra Z.

The Rokhlin property is further used in the computation of the K-theory of the C*-algebra D. which is done in Section 7. In fact, because of the Rokhlin property, the K-theory of the crossed product is the same as the K-theory of the fixed point algebra C_0^{γ}, and a result of Izumi [17] can be used to compute $K_*(C_0^{\gamma})$. See Proposition 7.3. Knowing the K-theory of D, and using the uniqueness of the trace, we can easily determine the Cuntz semigroup of D, which is given by $W(D) \cong \mathbb{Z}\left[\tfrac{1}{3}\right]_+ \sqcup (0, \infty)$. See Corollary 7.4.

Finally, the proof that the C*-algebra D is not isomorphic to its opposite algebra, which is done in Section 5, relies on two key points: uniqueness of

the tracial state on D, and the fact that the Gelfand-Naimark-Segal representation with respect to this tracial state yields a II_1 factor which is not isomorphic to its opposite algebra.

In Section 8 we give a generalization of the construction. In place of the number 3, we can use any odd prime p such that -1 is not a square mod p. The main point of doing this is that we get different K-theory and a different Cuntz semigroup.

We conclude with some open questions in Section 9.

2. Definitions

In this section, we give some definitions and known results which will play a key role in this paper.

Definition 2.1. A C*-algebra A is *exact* if for every short exact sequence of C*-algebras and homomorphisms

$$0 \longrightarrow J \longrightarrow B \longrightarrow B/J \longrightarrow 0,$$

the sequence of spatial tensor products

$$0 \longrightarrow A \otimes_{\min} J \longrightarrow A \otimes_{\min} B \longrightarrow A \otimes_{\min} (B/J) \longrightarrow 0$$

is exact.

The theory of exact C*-algebras was developed by Kirchberg in a series of papers, particularly [19], [20], and [21]. He proved a number of conditions equivalent to exactness for separable, unital C*-algebras. For example, he showed that exactness is equivalent to nuclear embeddability of the C*-algebra (Theorem 4.1 and the preceding comment in [20]). He also proved that the class of exact C*-algebra is closed under some of the standard operations on C*-algebras, like taking quotients, tensor products, direct limits, and crossed products by amenable locally compact groups (Proposition 7.1 of [21]). Nuclear C*-algebras are exact, so abelian C*-algebras, AF algebras, and group C*-algebras of amenable groups are all examples of exact C*-algebras.

Exactness is easily defined in the C*-algebra setting, but it is quite tricky to formulate an analog for von Neumann algebras. However, there are other properties, like having the order of projections determined by traces (as defined below), which have their natural setting in the class of von Neumann algebras.

Definition 2.2. Let A be a unital C*-algebra, and let $p, q \in A$ be projections. Then $p \precsim q$ if and only if there is $v \in A$ such that $p = vv^*$ and $v^*v \leq q$.

In a finite factor, the order on projections is determined by the trace. This means that if M is a finite factor with standard trace τ, and $e, f \in M$ are projections, then $e \precsim f$ if and only if $\tau(e) \leq \tau(f)$. In [2] Blackadar asked to what extent a similar comparison theory could be developed for simple unital C*-algebras. A simple unital C*-algebra A may have more than one tracial state, so we use the set $T(A)$ of all tracial states on A. The following definition is Blackadar's Second Fundamental Comparability Question for $\bigcup_{n=1}^{\infty} M_n(A)$. See 1.3.1 in [2].

Definition 2.3. Let A be a unital C*-algebra. Set $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$, using the usual embedding of $M_n(A)$ in $M_{n+1}(A)$ as the upper left corner. We say that *the order of projections over A is determined by traces* if whenever $p, q \in M_\infty(A)$ are projections with $q \neq 0$ such that $\tau(p) < \tau(q)$ for every τ in $T(A)$, then $p \precsim q$.

Unfortunately, one must sometimes use quasitraces instead of tracial states. (It is an open question whether every quasitrace is a trace.)

Definition 2.4 (4.2 of [27])**.** A (normalized) *quasitrace* on a unital C*-algebra is a not necessarily linear function $\tau \colon A \to \mathbb{C}$ satisfying:

(1) $\tau(1) = 1$.
(2) $\tau(xx^*) = \tau(x^*x) \geq 0$ for all $x \in A$.
(3) $\tau(a + ib) = \tau(a) + i\tau(b)$ for all $a, b \in A_{\mathrm{sa}}$.
(4) τ is linear on every commutative C*-subalgebra of A.
(5) For every $n \in \mathbb{N}$, the usual extension of τ to a function from $M_n(A)$ to $\mathbb{C}$ also satisfies (2)–(4).

Denote by $QT(A)$ the set of all quasitraces on A.

This terminology differs slightly from that in Definition II.1.1 of [3], which omits (1) (normalization) and, more importantly, (5).

Definition 2.5 (Definition 1.2 of [4])**.** Let A be a separable unital C*-algebra. We say that A is *approximately divisible* if for every $x_1, x_2, \ldots, x_n \in A$ and $\varepsilon > 0$, there exists a finite-dimensional C*-subalgebra B of A, containing the unit of A, such that:

(1) B has no commutative central summand, that is, B has no abelian central projection.
(2) $\|x_k y - y x_k\| < \varepsilon$ for $k = 1, 2, \ldots, n$ and all y in the unit ball of B.

Many standard simple C*-algebras are approximately divisible. For our purposes, we need:

Lemma 2.6. *The following C*-algebras are approximately divisible.*

(1) *Every infinite-dimensional simple unital AF algebra is approximately divisible.*
(2) *Let A and B be separable unital C*-algebras. If A is approximately divisible, then so is the tensor product $A \otimes B$, with any choice of tensor norm.*

Proof. Part (1) is contained in Proposition 4.1 of [4]. Part (2) is immediate. □

Proposition 2.7. *Let A be a finite approximately divisible simple separable unital exact C*-algebra. Then the order on projections over A is determined by traces.*

Proof. Corollary 3.9(b) of [4] implies that if $p, q \in M_\infty(A)$ are projections with $q \neq 0$ and $\tau(p) < \tau(q)$ for every quasitrace τ on A, then $p \precsim q$. It is proved in [14] that any quasitrace on a unital exact C*-algebra is a tracial state. It is now clear that if $\tau(p) < \tau(q)$ for every quasitrace τ on A, then $q \neq 0$. This verifies Definition 2.3. □

Remark 2.8. The paper [14] remains unpublished. It is therefore worth explaining how one can prove Proposition 2.7 using only published results. First, every quasitrace τ on A defines a state on $K_0(A)$ by using the extension of τ to $M_\infty(A)$ and the formula $\tau_*([p]) = \tau(p)$. The converse is also true (Theorem 3.3 of [5]). So it is enough to show that every state on $K_0(A)$ is actually induced by a tracial state on A. For a unital exact C*-algebra A, this is Corollary 9.18 of [15].

We follow Section 2 of [6] (not the original source) for the Cuntz semigroup.

Definition 2.9. Let A be a C*-algebra. Given $a, b \in M_\infty(A)_+$, we say that a is *Cuntz subequivalent* to b, denoted by $a \precsim b$, if there is a sequence $(v_n)_{n=1}^\infty$ in $M_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

Moreover, we say that a and b are *Cuntz equivalent*, denoted by $a \sim b$, if $a \precsim b$ and $b \precsim a$.

This is an equivalence relation, and we define $W(A)$ to be the set of equivalence classes, with the semigroup structure and partial order described in Section 2 of [6] and the set of equivalence classes with respect to this relation is a partially ordered semigroup. (See below.) This semigroup is an analog for positive elements of the Murray-von Neumann semigroup $V(A)$ of Murray-von Neumann equivalence classes of projections.

Definition 2.10. Given a C*-algebra A, we define the Cuntz semigroup $W(A)$ of A to be $M_\infty(A)_+/\sim$. We write $\langle a \rangle$ for the class of $a \in M_\infty(A)_+$. We define a semigroup operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle,$$

and a partial order by $\langle a \rangle \leq \langle b \rangle$ if and only if $a \precsim b$.

See Section 2 of [6] and the references there for proofs of the assertions made here.

In general, it is not easy to compute the Cuntz semigroup and it is not known how the Cuntz semigroup behaves with respect to tensor products or reduced free products.

If the C*-algebra A is stably finite, then two projections $p, q \in M_\infty(A)$ are Murray-von Neumann equivalent if and only if they are Cuntz equivalent, so the Murray-von Neumann semigroup $V(A)$ of a stably finite C*-algebra is always contained in its Cuntz semigroup $W(A)$. Moreover, In the case of a simple unital C*-algebra which is either exact and $\mathcal{Z}$-stable or an infinite dimensional AH algebra of slow dimension growth, it is shown in [6] that computing the Cuntz semigroup can be reduced to computing the Murray-von Neumann semigroup and the set of lower semicontinuous strictly positive affine functions on the tracial state space $T(A)$.

3. Uniqueness of the tracial state on some crossed products

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Consider the crossed product $A \rtimes_\alpha G = A \rtimes_{\alpha,\mathrm{r}} G$, with standard

unitaries u_g coming from the elements of G. The usual conditional expectation $E\colon A \rtimes_\alpha G \to A$ is given by $E\left(\sum_{g \in G} a_g u_g\right) = a_1$, where 1 is the identity of G. Any state ω on A gives rise to a state $\omega \circ E$ on $A \rtimes_\alpha G$.

Remark 3.1. Let the notation be as above. One easily checks that a state ψ on $A \rtimes_\alpha G$ is of the form $\omega \circ E$ for some state ω on A if and only if ψ is invariant under the dual action $\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(A \rtimes_\alpha G)$.

Lemma 3.2. *Let A be an infinite-dimensional unital C^*-algebra with a unique tracial state τ. Let α be an action of $\mathbb{Z}_n$ on A. Let π_τ be the Gelfand-Naimark-Segal representation of A corresponding to τ. Set $N_\tau = (\pi_\tau(A))'' \subseteq B(L^2(A, \tau))$. Denote by $\widetilde{\alpha}$ the extension of α to N_τ. Assume that the von Neumann algebra $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$ is a II_1 factor. Then there is a unique tracial state on $A \rtimes_\alpha \mathbb{Z}_n$.*

Proof. Let $E\colon A \rtimes_\alpha \mathbb{Z}_n \to A$ be the conditional expectation on the crossed product, as defined above. Then $\tau \circ E$ is a tracial state on $A \rtimes_\alpha \mathbb{Z}_n$. Suppose σ is a different tracial state on $A \rtimes_\alpha \mathbb{Z}_n$. Identifying $\widehat{\mathbb{Z}_n}$ with $\mathbb{Z}_n$, let $\widehat{\alpha}\colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_\alpha \mathbb{Z}_n)$ denote the dual action. Take $\mu = \frac{1}{n} \sum_{k=0}^{n-1} \sigma \circ \widehat{\alpha}_k$. Then μ is an $\widehat{\alpha}$-invariant tracial state, so by Remark 3.1 and uniqueness of τ, we have $\mu = (\mu|_A) \circ E = \tau \circ E$. Moreover,

$$\sigma \leq n \cdot (\tau \circ E).$$

We claim that σ extends to a normal tracial state $\widetilde{\sigma}$ on $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$. Denote by τ_0 the unique trace on $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$, so that $\tau_0|_{A \rtimes_\alpha \mathbb{Z}_n} = \tau \circ E$. Recall that the L^2-norm on $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$ is given by $\|y\|_2^2 = \tau_0(y^* y)$ for $y \in N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$. Fix $x \in N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$, and choose a sequence $(x_m)_{m \geq 0}$ in $A \rtimes_\alpha \mathbb{Z}_n$ such that $\lim_{m \to \infty} \|x_m - x\|_2 = 0$. Then for every $\varepsilon > 0$ there exists M such that $\|x_k - x_l\|_2 \leq \varepsilon/\sqrt{n}$ for $k, l > M$. Using the Cauchy-Schwarz inequality for the inner product corresponding to the tracial state σ at the first step, we get

$$\begin{aligned} |\sigma(x_l - x_k)|^2 &\leq \sigma((x_l - x_k)^*(x_l - x_k)) \leq n\big[\tau \circ E((x_l - x_k)^*(x_l - x_k))\big] \\ &= n\|x_l - x_k\|_2^2 \leq \varepsilon^2 \end{aligned}$$

for all $k, l > M$. So the sequence $(\sigma(x_m))_{m \geq 0}$ is Cauchy. Set $\widetilde{\sigma}(x) = \lim_{m \to \infty} \sigma(x_m)$. By a similar argument we can show that the definition of $\widetilde{\sigma}(x)$ does not depend on the choice of the sequence $(x_m)_{m \geq 0}$. This proves the claim.

Since $\widetilde{\sigma}$ is a normal trace on $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$ satisfying $\widetilde{\sigma} \leq n\tau_0$, there exists an element c in the center of $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$ such that:

- $0 \leq c \leq n$.
- $\tau_0(c) = 1$.
- $\widetilde{\sigma}(x) = \tau_0(xc)$ for every $x \in N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$.

The element c is not a scalar since $\widetilde{\sigma} \neq \tau_0$. This contradicts the assumption that $N_\tau \rtimes_{\widetilde{\alpha}} \mathbb{Z}_n$ is a factor. $\square$

4. The description of the C*-algebra $D = C_0 \rtimes_\gamma \mathbb{Z}_3$

In this section we describe our example of a C*-algebra D not isomorphic to its opposite algebra. It is the crossed product of a C*-algebra C_0 by an

action γ of the finite group $\mathbb{Z}_3$. The algebra C_0 is the tensor product $A_0 \otimes B_0$, where A_0 is a reduced free product C*-algebra and B_0 is a UHF algebra. The automorphism γ which generates our action of $\mathbb{Z}_3$ is a perturbation, by an inner automorphism, of an automorphism of the form $\alpha \otimes \beta$, with $\alpha \in \mathrm{Aut}(A_0)$ and $\beta \in \mathrm{Aut}(B_0)$. We start by describing the two C*-algebras A_0 and B_0, and the two automorphisms α and β.

Definition 4.1. Define $\varphi_n \colon M_{3^n} \to M_{3^{n+1}}$ by $\varphi_n(x) = \mathrm{diag}(x, x, x)$ for $x \in M_{3^n}$. Let B_0 be the UHF algebra obtained as the direct limit of the sequence $(M_{3^n}, \varphi_n)_{n \in \mathbb{N}}$. We identify M_{3^n} with the tensor product $M_3 \otimes M_3 \otimes \cdots \otimes M_3$ (n copies of M_3) and think of B_0 as the infinite tensor product C*-algebra $\bigotimes_1^\infty M_3$.

Notation 4.2. Denote by $B_0^{(n)}$ the image of the embedding $j_n \colon M_{3^n} \to B_0$, which we identify with

$$M_3 \otimes M_3 \otimes \cdots \otimes M_3 \otimes 1 \otimes 1 \otimes \cdots$$

(n copies of M_3). For $k \geq 1$, let $\pi_k \colon M_3 \to B_0$ be the map

$$\pi_k(x) = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes 1 \otimes \cdots ,$$

with x in position k, and denote by λ the shift endomorphism of B_0, determined by $\lambda(\pi_k(x)) = \pi_{k+1}(x)$ for all k and all $x \in M_3$. Let $(e_{i,j})_{i,j=1}^3$ the standard system of matrix units in M_3. Define unitaries $u, v \in B_0$ by

$$v = \pi_1\big(e^{2\pi i/3} e_{1,1} + e^{4\pi i/3} e_{2,2} + e_{3,3}\big)$$

and

$$u = \pi_1(e_{3,1})\lambda(v^*) + \pi_1(e_{1,2}) + \pi_1(e_{2,3}).$$

From the above data we construct an automorphism β of B_0 of outer period 3, that is, 3 is the least integer $m > 0$ such that β^m is inner. The following result is due to Connes (Proposition 1.6 of [7]), who proved it in the context of convergence in the strong operator topology by working with the L^2-norm. Below we simply outline the few changes from the von Neumann algebra setting to the C*-algebra setting.

Lemma 4.3 (Connes)**.** *Let the notation be as in Definition 4.1 and Notation 4.2. Define automorphisms $\beta_n \in \mathrm{Aut}(B_0)$ by*

$$\beta_n(x) = \mathrm{Ad}\big(u\lambda(u)\lambda^2(u)\dots\lambda^n(u)\big)(x)$$

for all $x \in B_0$. Then $\beta(x) = \lim_{n\to\infty} \beta_n(x)$ exists for every $x \in B_0$, and β is an outer automorphism of B_0 such that $\beta^3 = \mathrm{Ad}(v)$ and $\beta(v) = e^{2\pi i/3} v$.

Proof. For each fixed $k \geq 1$, and for any $j > 0$, the element $\lambda^{k+j}(u)$ commutes with all elements of $B_0^{(k)}$. Thus $\beta_{k+j}(x) = \beta_k(x)$ for any $j > 0$ and $x \in B_0^{(k)}$. Since $\bigcup_{k=0}^\infty B_0^{(k)}$ is dense in B_0, it follows that $\beta(y) = \lim_{n\to\infty} \beta_n(y)$ exists for all $y \in B_0$. This gives a homomorphism $\beta \colon B_0 \to B_0$, necessarily injective since B_0 is simple. Clearly $\beta(v) = uvu^* = e^{2\pi i/3} v$. Next, the proof of Proposition 1.6 of [7] shows that $\beta^3(y) = \mathrm{Ad}(v)(y)$ for every $y \in B_0$. Note that the equality $\beta^3(y) = \mathrm{Ad}(v)(y)$ implies that β is surjective, so β is an automorphism of B_0. Finally, β is outer because Proposition 1.6 of [7]

implies that it is outer on the type II_1 factor obtained from the Gelfand-Naimark-Segal construction using the unique tracial state on B_0. $\square$

The other algebra appearing in the construction of the crossed product is a reduced free product C*-algebra.

Definition 4.4. Define the C*-algebra A_0 to be the reduced free product

$$A_0 = C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} \mathbb{C}^3,$$

amalgamated over $\mathbb{C}$, taken with respect to the states given by Lebesgue measure μ on the first three factors and the state given by $\varphi(c_1, c_2, c_3) = \frac{1}{3}(c_1 + c_2 + c_3)$ on the last factor. For $k = 1, 2, 3$, let $\varepsilon_k \colon C([0,1]) \to A_0$ be the inclusion of the kth copy of $C([0,1])$ in the reduced free product. Let $u_0 = (e^{2\pi i/3},\, 1,\, e^{-2\pi i/3}) \in \mathbb{C}^3$, and regard u_0 as an element of A_0 via the obvious inclusion.

Lemma 4.5. *The C*-algebra A_0 of Definition 4.4 is unital, simple, separable, exact, and has a unique tracial state.*

Proof. It is trivial that A_0 is unital and separable. That it is simple and has a unique tracial state follows from several applications of the corollary on page 431 of [1]. (The notation is as in Proposition 3.1 of [1], and the required unitaries are easy to find.) Exactness follows from Theorem 3.2 of [12]. $\square$

Lemma 4.6. *Let the notation be as in Definition 4.4. Then there exists a unique automorphism $\alpha \in \mathrm{Aut}(A_0)$ such that, for all $f \in C([0,1])$,*

$$\alpha(\varepsilon_1(f)) = \varepsilon_2(f), \quad \alpha(\varepsilon_2(f)) = \varepsilon_3(f), \quad \textit{and} \quad \alpha(\varepsilon_3(f)) = \mathrm{Ad}(u_0)(\varepsilon_1(f)),$$

and such that $\alpha(u_0) = e^{-2\pi i/3} u_0$. Moreover, $\alpha^3 = \mathrm{Ad}(u_0)$.

Proof. We can identify $\mathbb{C}^3$ with the universal C*-algebra generated by a unitary of order 3, with the generating unitary taken to be u_0. The definition of $\alpha(u_0)$ is then legitimate because $\left(e^{-2\pi i/3} u_0\right)^3 = 1$. It is now obvious that if we replace A_0 by the full free product $C([0,1]) \star C([0,1]) \star C([0,1]) \star \mathbb{C}^3$ (taking u_0 and the values of ε_k to be in the full free product), then the definition gives an endomorphism α_0 of the full free product. Moreover, one checks immediately that $\alpha_0^3 = \mathrm{Ad}(u_0)$ on the full free product. Finally, using $\varphi(u_0) = 0$, it is easily checked that α_0 preserves the free product state $\mu \star \mu \star \mu \star \varphi$. (See Proposition 1.1 of [1] for the definition of the free product of two states.) Therefore α_0 descends to an automorphism α of $C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} \mathbb{C}^3$ such that $\alpha^3 = \mathrm{Ad}(u_0)$. $\square$

Definition 4.7. Let A_0, $u_0 \in A_0$, and $\alpha \in \mathrm{Aut}(A_0)$ be as in Definition 4.4. Let B_0 and $v \in B$ be as in Definition 4.1, and let $\beta \in \mathrm{Aut}(B_0)$ be as in Lemma 4.3. We define the C*-algebra C_0 by $C_0 = A_0 \otimes B_0$. (This tensor product is unambiguous since B_0 is nuclear.) Choose and fix a unitary $w \in C^*(u_0 \otimes v) \subseteq C_0$ such that $w^3 = (u_0 \otimes v)^*$ (see Lemma 4.8 below), and define $\gamma \in \mathrm{Aut}(C_0)$ by $\gamma = \mathrm{Ad}(w) \circ (\alpha \otimes \beta)$. We also write γ for the action of $\mathbb{Z}_3$ on C_0 generated by $\mathrm{Ad}(w) \circ (\alpha \otimes \beta)$. (See Lemma 4.8 below.)

Lemma 4.8. *Let the notation be as in Definition 4.7. Then there is a unitary* $w \in C^*(u_0 \otimes v) \subseteq C_0$ *such that* $w^3 = (u_0 \otimes v)^*$*, and the resulting* $\gamma \in \mathrm{Aut}(C_0)$ *satisfies* $\gamma^3 = \mathrm{id}_{C_0}$.

Proof. Since $\mathrm{sp}(u_0), \, \mathrm{sp}(v) \subseteq \{1, \, e^{2\pi i/3}, \, e^{-2\pi i/3}\}$, it immediately follows that $\mathrm{sp}(u_0 \otimes v)$ is finite. Therefore there is a unitary $w \in C^*(u_0 \otimes v)$ such that $w^3 = (u_0 \otimes v)^*$. Moreover, one checks that $(\alpha \otimes \beta)(u_0 \otimes v) = u_0 \otimes v$, whence $(\alpha \otimes \beta)(w) = w$. It is now a routine calculation to check that $\big(\mathrm{Ad}(w) \circ (\alpha \otimes \beta)\big)^3 = \mathrm{id}_{C_0}$. □

Lemma 4.9. *The C*-algebra* C_0 *of Definition 4.7 is exact and has a unique tracial state, namely the tensor product tracial state.*

Proof. This is true for A_0 by Lemma 4.5, and for B_0 because B_0 is a UHF algebra. For exactness, apply Proposition 7.1(ii) of [21], and for existence of a unique tracial state, apply Corollary 6.13 of [10]. □

Definition 4.10. Define the C*-algebra D by $D = C_0 \rtimes_\gamma \mathbb{Z}_3$.

5. The C*-algebra $D = C_0 \rtimes_\gamma \mathbb{Z}_3$ is not anti-isomorphic to itself

Definition 5.1. Let A be an arbitrary C*-algebra. The *opposite algebra* A^{op} of A is by definition the C*-algebra whose underlying Banach space and adjoint operation are the same as for A, while the product of x by y is equal to yx instead of xy.

Our main result is that the C*-algebra $D = (A_0 \otimes B_0) \rtimes_\gamma \mathbb{Z}_3$, given in Definition 4.10, is not isomorphic to its opposite algebra. We reduce the statement about the nonisomorphism of D and D^{op} to a statement about the nonisomorphism of the von Neumann algebra M associated to D (via the Gelfand-Naimark-Segal representation coming from the obvious tracial state of D) and M^{op}.

To this end, we first examine the weak operator closures of the images of A_0 and B_0 in the Gelfand-Naimark-Segal representations coming from their tracial states. These algebras were defined so that the statements in the following lemmas would hold.

Lemma 5.2. *Let* B_0 *be as in Definition 4.1, and let* $\beta \in \mathrm{Aut}(A_0)$ *be as in Lemma 4.3. Let* R_0 *be the weak operator closure of the image of* B_0 *under the Gelfand-Naimark-Segal representation coming from the unique tracial state of* B_0*, and let* $\widetilde{\beta}$ *be the automorphism of* R_0 *which extends* β*. Then* R_0 *is isomorphic to the hyperfinite factor* R *of type* II_1*. The isomorphism* $R_0 \cong R$ *can be chosen so that* $\widetilde{\beta}$ *becomes the automorphism of Proposition 1.6 of* [7]*, for the case* $p = 3$ *and* $\gamma = e^{2\pi i/3}$*. In particular, with* v *as in Notation 4.2, we have* $\widetilde{\beta}^3 = \mathrm{Ad}(v)$ *and* $\widetilde{\beta}(v) = e^{2\pi i/3} v$.

Proof. This is obvious from the definitions and from the proof of Proposition 1.6 of [7]. □

Lemma 5.3. *Let* A_0 *be as in Definition 4.4, and let* $\alpha \in \mathrm{Aut}(A_0)$ *be as in Lemma 4.6. Let* N_0 *be the weak operator closure of the image of* A_0 *under the Gelfand-Naimark-Segal representation coming from the unique tracial state*

of A_0 (Lemma 4.5), and let $\widetilde{\alpha}$ be the automorphism of N_0 which extends α. Then N_0 is a factor of type II_1 *isomorphic to the free product N of three copies of $L^{\infty}([0,1])$ (using Lebesgue measure) and the group von Neumann algebra $\mathcal{L}(\mathbb{Z}_3) \cong \mathbb{C}^3$ (using the tracial state φ of Definition 4.4). With*

$$\lambda_k \colon L^{\infty}([0,1]) \to L^{\infty}([0,1]) \star L^{\infty}([0,1]) \star L^{\infty}([0,1]) \star \mathcal{L}(\mathbb{Z}_3),$$

for $k = 1, 2, 3$, being the inclusion of the kth free factor, and with $u_0 \in \mathbb{C}^3 = \mathcal{L}(\mathbb{Z}_3)$ being as in Definition 4.4, the isomorphism $N_0 \cong N$ can be chosen so that, on N, for all $g \in L^{\infty}([0,1])$, we have

$$\widetilde{\alpha}(\lambda_1(g)) = \lambda_2(g), \quad \widetilde{\alpha}(\lambda_2(g)) = \lambda_3(g), \quad \textit{and} \quad \widetilde{\alpha}(\lambda_3(g)) = \mathrm{Ad}(u_0)(\lambda_1(g)),$$

and $\widetilde{\alpha}(u_0) = e^{-2\pi i/3} u_0$. Moreover, $(\widetilde{\alpha})^3 = \mathrm{Ad}(u_0)$.

Proof. This is obvious from the definitions. □

The nonisomorphism of the von Neumann algebra M of the next proposition and its opposite algebra was proved in [35]. We recall it here for the convenience of the reader, and refer to the original paper for the proof.

Proposition 5.4. *Let N, u_0, and $\widetilde{\alpha} \in \mathrm{Aut}(N)$ be as in Lemma 5.3, and let R, v, and $\widetilde{\beta} \in \mathrm{Aut}(R)$ be as in Lemma 5.2. Let w be as in Definition 4.7, regarded as an element of the von Neumann algebra tensor product $N \overline{\otimes} R$. Then $\mathrm{Ad}(w) \circ \big(\widetilde{\alpha} \otimes \widetilde{\beta}\big)$ generates an action $\widetilde{\gamma}$ of $\mathbb{Z}_3$ on $N \overline{\otimes} R$, and $M = (N \overline{\otimes} R) \rtimes_{\widetilde{\gamma}} \mathbb{Z}_3$ is a factor of type* II_1 *which is not isomorphic to its opposite von Neumann algebra.*

Proof. That $\widetilde{\gamma}$ generates an action of $\mathbb{Z}_3$ follows as in the proof of Lemma 4.8. (Or see [35].) That M is a factor is in Section 4 of [35], and $M \not\cong M^{\mathrm{op}}$ is Theorem 6.1 of [35]. □

Proposition 5.5. *The C*-algebra $D = C_0 \rtimes_{\gamma} \mathbb{Z}_3$ of Definition 4.10 is simple, separable, unital, exact, and has a unique tracial state.*

Proof. It is obvious that D is separable and unital.

The automorphism γ is outer, because β is outer (Lemma 4.3). Therefore $\gamma^2 = \gamma^{-1}$ is outer. Simplicity of D now follows from Theorem 3.1 of [23].

Exactness follows from Proposition 7.1(v) of [21] and Lemma 4.9.

To prove that D has a unique tracial state, apply Lemma 3.2, using Lemmas 4.9, 5.2, and 5.3, as well as Proposition 5.4, to verify its hypotheses. □

To reduce the nonisomorphism of the C*-algebra D and its opposite to the nonisomorphism of the von Neumann algebra M and its opposite, we use the unique tracial state defined on D and its associated Gelfand-Naimark-Segal representation.

Theorem 5.6. *The C*-algebra $D = C_0 \rtimes_{\gamma} \mathbb{Z}_3$ of Definition 4.10 is not isomorphic to its opposite algebra.*

Proof. Assume that there exists an isomorphism $\Phi \colon D \to D^{\mathrm{op}}$. Let τ be the unique tracial state on D (Proposition 5.5), and denote by τ^{op} the corresponding (unique) tracial state on D^{op}. Denote by π and π^{op} the Gelfand-Naimark-Segal representations of D and D^{op} associated to τ and τ^{op}. By

uniqueness of the tracial states, we have $\tau = \tau^{\mathrm{op}} \circ \Phi$, so π is unitarily equivalent to $\pi^{\mathrm{op}} \circ \Phi$. It follows that as von Neumann algebras $(\pi(D))''$ and $(\pi^{\mathrm{op}}(D^{\mathrm{op}}))''$ are isomorphic. As in the proof of Theorem 3.2 of [25], the algebra $(\pi(D))''$ is isomorphic to the factor M of Proposition 5.4, and $(\pi^{\mathrm{op}}(D^{\mathrm{op}}))'' \cong M^{\mathrm{op}}$. So $M \cong M^{\mathrm{op}}$, contradicting Proposition 5.4. □

6. The Rokhlin property

We obtain further properties of the C*-algebra $D = C_0 \rtimes_{\gamma} \mathbb{Z}_3$ by showing that γ has the Rokhlin property.

Definition 6.1. Let A be a unital C*-algebra and let $\sigma\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that σ has the *Rokhlin property* if for every finite set $F \subseteq A$ and every $\varepsilon > 0$, there exist nonzero mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\sigma_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) $\sum_{g \in G} e_g = 1$.

If an action $\sigma\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property, many properties of the C*-algebra A, such as real rank zero and stable finiteness, extend to the crossed product $A \rtimes_{\sigma} G$. So our next step is to show that the action $\gamma = \mathrm{Ad}(w) \circ (\alpha \otimes \beta)$ of Lemma 4.8 has the Rokhlin property. We first prove a Rokhlin-like property of order 3 for the automorphism β of Lemma 4.3.

Lemma 6.2. *Let B_0 be as in Definition 4.1, and let $\beta \in \mathrm{Aut}(A_0)$ be as in Lemma 4.3. Then for any $\varepsilon > 0$ and any finite set $F \subseteq B_0$ there exist projections p_1, p_2 and p_3 such that:*

(1) $\|\beta(p_i) - p_{i+1}\| < \varepsilon$ *for* $i = 1, 2$, *and* $\|\beta(p_3) - p_1\| < \varepsilon$.
(2) $\|p_j a - a p_j\| < \varepsilon$ *for all* $a \in F$ *and all* $j \in \{1, 2, 3\}$.
(3) $p_1 + p_2 + p_3 = 1$.

Proof. Adopt the notation of Notation 4.2 and Lemma 4.3. Since $B_0 = \varinjlim_n B_0^{(n)}$, we need only consider finite subsets F such that there is n with $F \subseteq B_0^{(n)}$. Set

$$p_1 = \pi_{n+1}(e_{1,1}), \quad p_2 = \pi_{n+1}(e_{3,3}), \quad \text{and} \quad p_3 = \pi_{n+1}(e_{2,2}).$$

Obviously $p_1 + p_2 + p_3 = 1$. Since p_1, p_2, and p_3 commute with every element in F, condition (2) is verified. To check condition (1), we need to compute $\beta(p_j)$ for $j \in \{1, 2, 3\}$. Note that if $0 \leq k \leq n-2$ or $k \geq n+1$, then $\lambda^k(u)$ commutes with p_j for $j \in \{1, 2, 3\}$. We have

$$\mathrm{Ad}(\lambda^n(u))(p_1) = \lambda^n\big((e_{3,1} \otimes v^* \otimes 1 \otimes \cdots)(e_{1,1} \otimes 1 \otimes \cdots)(e_{3,1} \otimes v^* \otimes 1 \otimes \cdots)^*\big) = p_2,$$
$$\mathrm{Ad}(\lambda^n(u))(p_2) = \lambda^n\big((e_{2,3} \otimes 1 \otimes 1 \otimes \cdots)(e_{3,3} \otimes 1 \otimes \cdots)(e_{2,3} \otimes 1 \otimes 1 \otimes \cdots)^*\big) = p_3,$$
$$\mathrm{Ad}(\lambda^n(u))(p_3) = \lambda^n\big((e_{1,2} \otimes 1 \otimes 1 \otimes \cdots)(e_{2,2} \otimes 1 \otimes \cdots)(e_{1,2} \otimes 1 \otimes 1 \otimes \cdots)^*\big) = p_1.$$

Also,

$$\lambda^{n-1}(u) = \pi_n(e_{3,1})\pi_{n+1}(v^*) + \pi_n(e_{1,2}) + \pi_n(e_{2,3}),$$

and v^* commutes with $e_{1,1}$, $e_{2,2}$, and $e_{3,3}$, so $\lambda^{n-1}(u)$ commutes with p_1, p_2, and p_3. Therefore

$$\beta(p_1) = \mathrm{Ad}(\lambda^{n-1}(u)\lambda^n(u))(p_1) = p_2,$$
$$\beta(p_2) = \mathrm{Ad}(\lambda^{n-1}(u)\lambda^n(u))(p_2) = p_3,$$
$$\beta(p_3) = \mathrm{Ad}(\lambda^{n-1}(u)\lambda^n(u))(p_3) = p_1.$$

So p_1, p_2, and p_3 verify condition (1). □

Proposition 6.3. *The action $\gamma\colon \mathbb{Z}_3 \to \mathrm{Aut}(C_0)$ of Definition 4.7 has the Rokhlin property.*

Proof. Recall that γ is generated by the automorphism (also called γ) $\mathrm{Ad}(w)\circ(\alpha\otimes\beta)$ of $C_0 = A_0 \otimes B_0$.

We first claim that for any finite subset $F \subseteq A_0 \otimes B_0$ and any $\varepsilon > 0$, there are projections $e_1, e_2, e_3 \in A_0 \otimes B_0$ such that:

(1) $\|(\alpha\otimes\beta)(e_j) - e_{j+1}\| < \varepsilon$ for $j = 1, 2$, and $\|(\alpha\otimes\beta)(e_3) - e_1\| < \varepsilon$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for all $a \in F$ and all $j \in \{1, 2, 3\}$.
(3) $e_1 + e_2 + e_3 = 1$.

A standard approximation argument shows that it suffices to choose a subset $G \subseteq A_0 \otimes B_0$ which generates a dense *-subalgebra of $A_0 \otimes B_0$, and consider only finite subsets $F \subseteq G$. Thus, we may assume that there are finite subsets $S \subseteq A_0$ and $T \subseteq B_0$ such that

$$F = \big\{a \otimes b\colon a \in S \text{ and } b \in T\big\}.$$

Set $M = \sup\{\|a\|\colon a \in S\}$. Apply Lemma 6.2 with $\varepsilon/(M+1)$ in place of ε, obtaining projections $p_1, p_2, p_3 \in B_0$ as there. Set $e_j = 1 \otimes p_j$ for $j \in \{1, 2, 3\}$. Conditions (1) and (3) above are immediate. For (2), let $a \in S$ and $b \in T$, and let $j \in \{1, 2, 3\}$. Then

$$\|(1\otimes p_j)(a\otimes b) - (a\otimes b)(1\otimes p_j)\| = \|a\|\|p_j b - b p_j\| \leq M\cdot\varepsilon/(M+1) < \varepsilon.$$

This proves the claim.

To show that the action $\gamma = \mathrm{Ad}(w)\circ(\alpha\otimes\beta)$ has the Rokhlin property, let $F \subseteq C_0$ be finite and let $\varepsilon > 0$. It suffices to find projections $e_1, e_2, e_3 \in C_0$ satisfying Conditions (1)–(3) in the claim above with γ in place of $\alpha\otimes\beta$. Choose $e_1, e_2, e_3 \in C_0$ as in the claim with $F \cup \{w\}$ in place of F and $\frac{1}{2}\varepsilon$ in place of ε. The analogs of Conditions (2) and (3) above are immediate. For (1), estimate

$$\begin{aligned}\|\gamma(e_1) - e_2\| &= \|(\alpha\otimes\beta)(e_1) - w^* e_2 w\| \leq \|(\alpha\otimes\beta)(e_1) - e_2\| + \|e_2 - w^* e_2 w\| \\ &= \|(\alpha\otimes\beta)(e_1) - e_2\| + \|w e_2 - e_2 w\| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.\end{aligned}$$

Similarly $\|\gamma(e_2) - e_3\| < \varepsilon$ and $\|\gamma(e_3) - e_1\| < \varepsilon$. □

Remark 6.4. Let A be a simple infinite-dimensional finite unital C*-algebra, and let $\sigma\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A satisfying the Rokhlin property. Assume that A has a unique tracial state. Then Corollary 4.10 of [24] implies that $A \rtimes_\sigma G$ also has a unique tracial state. Thus, the fact that γ has the Rokhlin property, combined with Lemma 4.9, provides an alternative proof of the assertion in Proposition 5.5 that D has a unique tracial state.

We now use the Rokhlin property for γ to obtain further properties of D. We recall the Jiang-Su algebra Z, from Theorem 2.9 of [18]. It is a simple separable unital C*-algebra which is not of type I and which has a unique tracial state.

Proposition 6.5. *The C*-algebra $D = C_0 \rtimes_{\gamma} \mathbb{Z}_3$ of Definition 4.10 is approximately divisible, stably finite, and has real rank zero and stable rank one. The order on projections over D is determined by the unique tracial state on D, in the sense of Definition 2.3. Moreover, D tensorially absorbs the 3^{∞} UHF algebra B_0 and the Jiang-Su algebra Z.*

Proof. The C*-algebra D is stably finite since it is simple and has a (necessarily faithful) tracial state.

The same applies to A_0. Since B_0 is a UHF algebra, Corollary 6.6 in [26] implies that $\mathrm{tsr}(A_0 \otimes B_0) = 1$. Since γ has the Rokhlin property, Proposition 4.1(1) in [24] now implies that $\mathrm{tsr}(D) = 1$.

Since any quasitrace on a unital exact C*-algebra is a tracial state [14], Lemma 4.5 implies that the projections in A_0 distinguish the quasitraces. Theorem 7.2 in [27] now implies that $A_0 \otimes B_0$ has real rank zero, and Proposition 4.1(2) in [24] implies that D has real rank zero.

Lemma 2.6(1) implies that B_0 is approximately divisible, so Lemma 2.6(2) implies that $A_0 \otimes B_0$ is approximately divisible, and Proposition 4.5 in [24] (or Corollary 3.4(2) in [16]) implies that D is approximately divisible.

Proposition 2.7 now implies that the order on projections over D is determined by traces.

For the absorption properties, first note that $C_0 \otimes B_0 = A_0 \otimes B_0 \otimes B_0 \cong A_0 \otimes B_0$. Thus C_0 absorbs B_0. It now follows from Corollary 3.4(1) of [16] that D absorbs B_0. Now D absorbs Z because, by Corollary 6.3 of [18], the algebra B_0 absorbs Z. □

Remark 6.6. Since the paper [14] remains unpublished, we point out that one can instead use Corollary 9.18 of [15]. The hypothesis on quasitraces in Theorem 7.2 in [27] enters the proof through Theorem 5.2(b) there (which is used in the proof of Lemma 7.1 there). However, all that is really needed is that a certain kind of dimension function on projections comes from a quasitrace, and Corollary 9.18 of [15] implies that, on an exact C*-algebra, it in fact comes from a tracial state.

7. The K-theory of the crossed product $D = C_0 \rtimes_{\gamma} \mathbb{Z}_3$

In this section, we compute the K-theory of the algebra

$$D = C_0 \rtimes_{\gamma} \mathbb{Z}_3 = (A_0 \otimes B_0) \rtimes_{\mathrm{Ad}(w) \circ (\alpha \otimes \beta)} \mathbb{Z}_3.$$

We start with the K-theory of A_0 and B_0.

The K-theory of B_0 is well known, and is given by $K_0(B_0) \cong \mathbb{Z}\left[\frac{1}{3}\right]$ and $K_1(B) = 0$. In the first isomorphism, $[1]$ is sent to 1.

Lemma 7.1. *Let A_0 be as in Definition 4.4. Then the inclusion $\eta \colon \mathbb{C}^3 \to A_0$ of the last free factor is a KK-equivalence. In particular, $K_1(A_0) = 0$ and $K_0(A_0) \cong \mathbb{Z}^3$, generated by the classes of the images of the projections*

$$r_1 = (1, 0, 0), \quad r_2 = (0, 1, 0), \quad \textit{and} \quad r_3 = (0, 0, 1)$$

in $\mathbb{C}^3$.

Proof. First, the inclusion $\mathbb{C} \to C([0,1])$ is clearly a homotopy equivalence. By taking the full free product of homotopies, we find that

$$\mathbb{C} = \mathbb{C} \star \mathbb{C} \star \mathbb{C} \to C([0,1]) \star C([0,1]) \star C([0,1])$$

(full free products) is a homotopy equivalence. Therefore

$$\mathbb{C}^3 = \mathbb{C} \star \mathbb{C}^3 \to C([0,1]) \star C([0,1]) \star C([0,1]) \star \mathbb{C}^3$$

is a homotopy equivalence. Theorem 4.1 of of [13] (see the introduction to that paper for the notation A and A_r used there) implies that

$$C([0,1]) \star C([0,1]) \star C([0,1]) \star \mathbb{C}^3 \to C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} \mathbb{C}^3 = A_0$$

is a KK-equivalence. (Note that "K-equivalence" in [13] is what is usually called KK-equivalence; see Section 6 of [31].) ☐

Corollary 7.2. *Let $C_0 = A_0 \otimes B_0$ be as in Definition 4.7. Then*

$$K_1(C_0) = 0 \quad \textit{and} \quad K_0(C_0) \cong \mathbb{Z}\left[\tfrac{1}{3}\right]^3.$$

The inclusion $\mathbb{C}^3 \to A_0$ of the last free factor in A_0 gives an inclusion $\mathbb{C}^3 \otimes B_0 \to C_0$, and this map is an isomorphism on K-theory.

Proof. We can get this by tensoring the result of Lemma 7.1 with $\mathrm{id}_{M_{3^n}}$ and taking direct limits. However, it is easier to apply the Künneth formula [30]; this is valid because B_0 is in the bootstrap category of [30]. ☐

Proposition 7.3. *Let $D = C_0 \rtimes_\gamma \mathbb{Z}_3$ be as in Definition 4.10. Then*

$$K_0(D) \cong \mathbb{Z}\left[\tfrac{1}{3}\right] \quad \textit{and} \quad K_1(D) = 0.$$

The first isomorphism sends $[1]$ to 1, and is an isomorphism of ordered groups. Letting τ be the tracial state on D as in Proposition 5.5, the map $\tau_*\colon K_0(D) \to \mathbb{R}$ corresponds to the inclusion of $\mathbb{Z}\left[\tfrac{1}{3}\right]$ in $\mathbb{R}$.

Proof. We use the notation of Definition 4.1, Lemma 4.3, Definition 4.4, Lemma 4.6, and Definition 4.7.

We begin by computing the K-theory of the fixed point algebra C_0^γ of C_0 under the action γ. Since C_0 is simple and unital, and γ has the Rokhlin property (Proposition 6.3), we can apply Theorem 3.13 in [17] to conclude, first, that the inclusion of C_0^γ in C_0 is injective on K-theory, and, second, that its range is

$$\bigcap_{m=0}^{2} \ker(\mathrm{id} - K_*(\gamma^m)).$$

We can ignore the term for $m = 0$. Since

$$\mathrm{id} - K_*(\gamma^2) = K_*(\gamma^2) \circ (-(\mathrm{id} - K_*(\gamma)))$$

and $K_*(\gamma^2)$ is injective, we can also ignore the term for $m = 2$. Thus, $K_*(C_0^\gamma) \cong \ker(\mathrm{id} - K_*(\gamma))$.

It immediately follows that $K_1(C_0^\gamma) = 0$.

We claim that $K_0(\gamma)\colon \mathbb{Z}\left[\tfrac{1}{3}\right]^3 \to \mathbb{Z}\left[\tfrac{1}{3}\right]^3$ is given by permuting the coordinates: $(\eta_1, \eta_2, \eta_3) \mapsto (\eta_3, \eta_1, \eta_2)$. Since $\mathrm{Ad}(w)$ is trivial on K-theory, it

suffices to show that this formula is correct for $K_0(\alpha \otimes \beta)$. Since every automorphism, in particular β, of B_0 is trivial on K-theory, Corollary 7.2 implies that it is enough to show that this formula is correct for $K_0(\alpha)\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ as in Lemma 7.1. Define projections in $\mathbb{C}^3 \subseteq A_0$ by

$$r_1 = (1, 0, 0), \quad r_2 = (0, 1, 0), \quad \text{and} \quad r_3 = (0, 0, 1).$$

Then the unitary u_0 of Definition 4.4 is given by $u_0 = e^{2\pi i/3} r_1 + r_2 + e^{-2\pi i/3} r_3$. Since it generates $\mathbb{C}^3$ and $\alpha(u_0) = e^{-2\pi i/3} u_0 = r_1 + e^{-2\pi i/3} r_2 + e^{2\pi i/3} r_3$, we must have $\alpha(r_1) = r_3$, $\alpha(r_2) = r_1$, and $\alpha(r_3) = r_2$. The desired formula is now immediate.

We now know that $\mathrm{id} - K_0(\gamma)$ is given by the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

The map $\eta \mapsto (\eta, \eta, \eta)$ is an isomorphism from $\mathbb{Z}\left[\frac{1}{3}\right]$ to $\ker(\mathrm{id} - K_0(\gamma))$ which sends 1 to $(1, 1, 1) = [1]$. This completes the computation of $K_*(C_0^\gamma)$.

Let $z \in C_0 \rtimes_\gamma \mathbb{Z}_3$ be the standard unitary corresponding to the usual generator of $\mathbb{Z}_3$, and let $p \in C_0 \rtimes_\gamma \mathbb{Z}_3$ be the projection $p = \frac{1}{3}(1 + z + z^2)$. The Proposition, Corollary, and proof of the Corollary in [28] imply that C_0^γ is isomorphic to the corner $p(C_0 \rtimes_\gamma \mathbb{Z}_3)p$. Since $C_0 \rtimes_\gamma \mathbb{Z}_3$ is simple, this corner is full, and its inclusion is an isomorphism on K-theory. Thus, $K_1(C_0 \rtimes_\gamma \mathbb{Z}_3) = 0$ and $K_0(C_0 \rtimes_\gamma \mathbb{Z}_3) \cong \mathbb{Z}\left[\frac{1}{3}\right]$, with $[p]$ corresponding to $1 \in \mathbb{Z}\left[\frac{1}{3}\right]$. Let $\widehat{\gamma}\colon \mathbb{Z}_3 \to \mathrm{Aut}(C_0 \rtimes_\gamma \mathbb{Z}_3)$ denote the generator of the dual action. Then $p + \widehat{\gamma}(p) + \widehat{\gamma}^2(p) = 1$. Moreover, $K_0(\widehat{\gamma})$ is an automorphism of $\mathbb{Z}\left[\frac{1}{3}\right]$ which fixes the nonzero element $[1]$, so we must have $K_0(\widehat{\gamma}) = \mathrm{id}$. Therefore $[1] = 3[p]$. Since multiplication by 3 is an automorphism of $\mathbb{Z}\left[\frac{1}{3}\right]$, it follows that there is an isomorphism $K_0(C_0 \rtimes_\gamma \mathbb{Z}_3) \cong \mathbb{Z}\left[\frac{1}{3}\right]$ which sends $[1]$ to $1 \in \mathbb{Z}\left[\frac{1}{3}\right]$.

The map $\tau_*\colon K_0(D) \to \mathbb{R}$ is a homomorphism which sends $[1]$ to $1 \in \mathbb{R}$. There is a unique homomorphism $\mathbb{Z}\left[\frac{1}{3}\right] \to \mathbb{R}$ which sends 1 to 1, namely the inclusion. Therefore the isomorphism $K_0(C_0 \rtimes_\gamma \mathbb{Z}_3) \cong \mathbb{Z}\left[\frac{1}{3}\right]$ must send τ_* to the inclusion. Since the order on projections over D is determined by the tracial states (Proposition 6.5), it follows that this isomorphism is an order isomorphism for the usual order on $\mathbb{Z}\left[\frac{1}{3}\right]$. □

Since D is simple and tensorially absorbs the Jiang-Su algebra, we can also determine the Cuntz semigroup of the C*-algebra D. (See Definitions 2.9 and 2.10.)

Corollary 7.4. *Let $D = C_0 \rtimes_\gamma \mathbb{Z}_3$ be as in Definition 4.10. The Cuntz semigroup of D is given by $W(D) \cong \mathbb{Z}\left[\frac{1}{3}\right]_+ \sqcup (0, \infty)$. Here, $\mathbb{Z}\left[\frac{1}{3}\right]_+$ is the set of nonnegative elements in $\mathbb{Z}\left[\frac{1}{3}\right]$. Addition and the order on each part of the disjoint union are the usual ones. If $\eta \in \mathbb{Z}\left[\frac{1}{3}\right]_+$ and $\widetilde{\eta} \in (0, \infty)$ is the corresponding element of $(0, \infty)$, and if $\mu \in (0, \infty)$, then $\eta \leq \mu$ if and only if $\widetilde{\eta} < \mu$ in $(0, \infty)$, while $\eta \geq \mu$ if and only if $\widetilde{\eta} \geq \mu$ in $(0, \infty)$. Moreover, $\eta + \mu = \eta_0 + \mu$, the right hand side being computed in $(0, \infty)$.*

That is, in the order on $\mathbb{Z}\big[\frac{1}{3}\big]_{+} \sqcup (0,\infty)$, elements of $(0,\infty)$ are less than the corresponding elements of $\mathbb{Z}\big[\frac{1}{3}\big]_{+}$, while the other order relations are the "obvious" ones.

Proof of Corollary 7.4. Since D is simple, unital, exact, and stably finite (this all follows from Proposition 5.5), and Z-stable (this follows from Proposition 6.5), we can apply Corollary 5.7 of [6]. The group $\widetilde{W}(D)$ which appears there is defined on page 195 of [6], and it is the disjoint union of two pieces: $V(D)$, the semigroup of Murray-von Neumann equivalence classes of projections in $M_{\infty}(D)$, and $\mathrm{LAff}_b(T(D))^{++}$, the set of bounded, strictly positive, lower semicontinuous, affine functions on $T(D)$, the tracial state space of D. We get $V(D) \cong \mathbb{Z}\big[\frac{1}{3}\big]_{+}$ from the K-theory computation (Proposition 7.3) and because the order on projections over D is determined by the tracial states (Proposition 6.5). Since $T(D)$ consists of just one point, it is immediate that $\mathrm{LAff}_b(T(D))^{++} \cong (0,\infty)$. The proof is completed by observing that the order and semigroup operation in the statement match those defined on $\widetilde{W}(D)$ in [6]. $\square$

Remark 7.5. We would like to be able to say that the algebra D of Definition 4.10 satisfies the Universal Coefficient Theorem of [29]. Making allowances for the fact that it is not nuclear, the algebra A_0 does, because Lemma 7.1 implies that A_0 is KK-equivalent to $\mathbb{C}^3$. Therefore also C_0 satisfies the Universal Coefficient Theorem. In general, the Universal Coefficient Theorem does not pass to crossed products by finite groups. We almost get the Universal Coefficient Theorem for D from Corollary 3.9 of [24]. However, that corollary assumes nuclearity. The proof is based on results in [11], and nuclearity plays an important role in the proofs.

8. A generalization of the C*-algebra D

In place of $\mathbb{Z}_3$ in the construction of this paper, one can use $\mathbb{Z}_p$ for any odd prime p such that -1 is not a square mod p. The resulting algebra has different K-theory and Cuntz semigroup. We describe the differences in the construction. We will use the same notation as in the previous sections for the algebras we have constructed.

Throughout this section, p is a fixed odd prime, usually such that -1 is not a square mod p.

Definition 8.1 (Compare with Definitions 4.4 and 4.1)**.** Let $p \neq 2$ be a prime number. Define the C*-algebra A_0 to be the reduced free product of p copies of $C([0,1])$ and the C*-algebra $\mathbb{C}^p$:

$$A_0 = C([0,1]) \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} \cdots \star_{\mathrm{r}} C([0,1]) \star_{\mathrm{r}} \mathbb{C}^p,$$

amalgamated over $\mathbb{C}$, taken with respect to the states given by Lebesgue measure μ on the first p factors and the state given by

$$\omega(c_1, c_2, \ldots, c_p) = \tfrac{1}{p}(c_1 + c_2 + \cdots + c_p)$$

on the last factor. As in Section 4, for $k = 1, 2, \ldots, p$ we denote by $\varepsilon_k \colon C([0,1]) \to A$ the inclusion of the k-th copy of $C([0,1])$ in the C*-algebra A_0. Set

$$u_0 = \left(e^{2\pi i/p}, 1, e^{2(p-1)\pi i/p}, e^{2(p-2)\pi i/p}, \ldots, e^{4\pi i/p}\right) \in \mathbb{C}^p,$$

and regard u_0 as a unitary in A_0 via the obvious inclusion.

Define $\varphi_n \colon M_{p^n} \to M_{p^{n+1}}$ by $\varphi(x) = \operatorname{diag}(x, x, \ldots, x)$ for $x \in M_{p^n}$. Denote by B_0 the UHF algebra obtained as the direct limit of the sequence $(M_{p^n}, \varphi_n)_{n \in \mathbb{N}}$. We identify M_{p^n} with $\bigotimes_1^n M_p$, and B_0 with $\bigotimes_1^\infty M_p$.

The same argument as for Lemma 4.6 shows that there exists a unique automorphism $\alpha \in \operatorname{Aut}(A_0)$ such that for all $f \in C([0,1])$ and for $k = 1, 2, \ldots, p-1$, we have $\alpha(\varepsilon_k(f)) = \varepsilon_{k+1}(f)$, and $\alpha(\varepsilon_p(f)) = \operatorname{Ad}(u_0)(\varepsilon_1(f))$, and such that $\alpha(u_0) = e^{-2\pi i/p} u_0$. Moreover, $\alpha^p = \operatorname{Ad}(u_0)$.

Let $\pi_k \colon M_p \to B_0$ and $\lambda \colon B_0 \to B_0$ be defined by the same formulas as in Notation 4.2. Denote by $(e_{i,j})_{i,j=1}^p$ the standard system of matrix units in M_p. Define unitaries $v, u \in B_0$ by

$$v = \pi_1\left(\sum_{k=1}^{p} e^{2\pi i k/p} e_{k,k}\right) \quad \text{and} \quad u = \pi_1(e_{p,1})\lambda(v^*) + \sum_{k=1}^{p-1} \pi_1(e_{k,\,k+1}).$$

The argument of Lemma 4.3 shows that

$$\beta(x) = \lim_{n \to \infty} \operatorname{Ad}\big(u\lambda(u) \cdots \lambda^n(u)\big)(x)$$

defines an outer automorphism of B_0 such that $\beta^p = \operatorname{Ad}(v)$ and $\beta(v) = e^{2\pi i/p} v$.

Definition 8.2 (Compare with Definitions 4.7 and 4.10)**.** Using the notation of the previous definitions, set $C_0 = A_0 \otimes B_0$. Let $w \in C^*(u_0 \otimes v)$ be a p-th root of $u_0 \otimes v$, and set $\gamma = \operatorname{Ad}(w) \circ (\alpha \otimes \beta)$. Define the C*-algebra D by $D = C_0 \rtimes_\gamma \mathbb{Z}_p$.

The argument of the proof of Lemma 4.8 shows that w exists and that γ^p really is id_{C_0}. Provided that -1 is not a square mod p, the C*-algebra D defined here has all the same properties as the C*-algebra D in the previous sections, except with 3 replaced by p. To prove that $D \not\cong D^{\mathrm{op}}$, we will need some terminology and results for automorphisms of factors.

Definition 8.3. Let M be a II_1 factor with separable predual. An automorphism φ of M is *approximately inner* if there exists a sequence of unitaries $(u_n)_{n \in \mathbb{N}}$ in M such that $\varphi = \lim_{n \to \infty} \operatorname{Ad}(u_n)$, in the sense of pointwise convergence in $\|\cdot\|_2$. Moreover, φ is said to be *centrally trivial* if $\lim_{n \to \infty} \|\varphi(x_n) - x_n\|_2 = 0$ for every central sequence $(x_n)_{n \in \mathbb{N}}$ in M, that is, any bounded sequence that asymptotically commutes in $\|\cdot\|_2$ with every element of M. Further, let $\operatorname{Inn}(M)$ denote the group of inner automorphisms of M. Then the *Connes invariant* $\chi(M)$ is the subgroup of the outer automorphism group $\operatorname{Aut}(M)/\operatorname{Inn}(M)$ obtained as the center of the image of the approximately inner automorphisms. (See Connes [7]. Connes uses centralizing sequences, which in general is the correct choice, but for a II_1 factor these are the same as central sequences.)

If M is isomorphic to its tensor product with the hyperfinite II_1 factor, $\chi(M)$ is the image in $\mathrm{Aut}(M)/\mathrm{Inn}(M)$ of the intersection of the approximately inner automorphisms with the centrally trivial automorphisms. See [7].

The following result belongs to the theory of cocycle conjugacy, but we have been unable to find a reference.

Lemma 8.4. *Let M be a factor, let $n \in \mathbb{N}$, write the elements of $\mathbb{Z}_n$ as $0, 1, \ldots, n-1$, and let $\alpha, \beta \colon \mathbb{Z}_n \to \mathrm{Aut}(M)$ be actions such that there is a unitary $y \in M$ with $\beta_1 = \mathrm{Ad}(y) \circ \alpha_1$. Then there is an isomorphism $\varphi \colon M \rtimes_\beta \mathbb{Z}_n \to M \rtimes_\alpha \mathbb{Z}_n$ which intertwines the dual actions, that is, for all $l \in \widehat{\mathbb{Z}_n}$ we have $\varphi \circ \widehat{\beta}_l = \widehat{\alpha}_l \circ \varphi$.*

Proof. For $k \in \mathbb{Z}$ we write $\alpha_k = \alpha_1^k$ and $\beta_k = \beta_1^k$. (This agrees with the notation in the statement when $k \in \{0, 1, \ldots, n-1\}$.) For $k \in \mathbb{N}$ define a unitary in M by

$$y_k = y\alpha_1(y)\alpha_2(y)\cdots\alpha_{k-1}(y).$$

Set $y_0 = 1$, and define $y_k = \alpha_k(y_{-k})$ for $k < 0$. Then one easily checks that $\mathrm{Ad}(y_k) \circ \alpha_k = \beta_k$ for all $k \in \mathbb{Z}$, and moreover that $y_j \alpha_j(y_k) = y_{j+k}$ for all $j, k \in \mathbb{Z}$.

Since $\alpha_n = \beta_n = \mathrm{id}_M$, and since M is a factor, y_n is a scalar, and there is a scalar ζ with $|\zeta| = 1$ such that $\zeta^n = y_n$. For $k \in \mathbb{Z}$ define $z_k = \zeta^k y_{-k}$. Then z_k is a unitary, and we have $\mathrm{Ad}(z_k) \circ \alpha_k = \beta_k$ for all $k \in \mathbb{Z}$ and $z_j \alpha_j(z_k) = z_{j+k}$ for all $j, k \in \mathbb{Z}$. Moreover, $z_j = z_k$ whenever n divides $j - k$.

Let $u_0, u_1, \ldots u_{n-1} \in M \rtimes_\alpha \mathbb{Z}_n$ be the standard unitaries, so that for $a \in M \subset M \rtimes_\alpha \mathbb{Z}_n$ we have $u_k a u_k^* = \alpha_k(a)$ and

$$M \rtimes_\alpha \mathbb{Z}_n = \left\{ \sum_{k=0}^{n-1} a_k u_k \colon a_0, a_1, \ldots a_{n-1} \in M \right\}.$$

Similarly let $v_0, v_1, \ldots v_{n-1}$ be the standard unitaries in $M \rtimes_\beta \mathbb{Z}_n$. Then there is a unique linear bijection $\varphi \colon M \rtimes_\alpha \mathbb{Z}_n \to M \rtimes_\beta \mathbb{Z}_n$ such that $\varphi(a v_k) = a z_k u_k$ for $a \in M$ and $k = 0, 1, \ldots, n-1$. One checks, using the properties of the z_k, that φ is a homomorphism. It is immediate that φ intertwines the dual actions. ∎

Proposition 8.5. *Let p be any odd prime such that -1 is not a square mod p. The C^*-algebra $D = C_0 \rtimes_\gamma \mathbb{Z}_p$ of Definition 8.2 is simple, separable, unital, exact, and not isomorphic to its opposite algebra. The algebra D tensorially absorbs the p^∞ UHF algebra B_0 and the Jiang-Su algebra Z. Moreover, D is approximately divisible, stably finite, has real rank zero and stable rank one, and has a unique tracial state which determines the order on projections over D. Also,*

$$K_0(D) \cong \mathbb{Z}\left[\tfrac{1}{p}\right] \quad \text{and} \quad K_1(D) = 0,$$

where the first isomorphism sends $[1]$ to 1, is an isomorphism of ordered groups, and sends the map on $K_0(D)$ induced by the tracial state to the inclusion of $\mathbb{Z}\left[\tfrac{1}{p}\right]$ in $\mathbb{R}$. Finally, the Cuntz semigroup of D is given by

$$W(D) \cong \mathbb{Z}\left[\tfrac{1}{p}\right]_+ \sqcup (0, \infty),$$

with the order and semigroup operation being the obvious analogs of those defined in Corollary 7.4.

Proof. The main differences are in the computation of $K_0(D)$ and in the proof that $D \not\cong D^{\mathrm{op}}$. In particular, the same proof as for Proposition 5.5 shows that D is simple, separable, unital, exact, and has a unique tracial state. (Note that, if γ^k were inner for any $k \in \{1, 2, \ldots, p-1\}$, then it would follow that γ is inner, since p is prime.) The same proof as for Proposition 6.5 shows that D is approximately divisible, stably finite, has real rank zero and stable rank one, and tensorially absorbs B_0 and Z, as well as that the order on projections over D is determined by traces. Once $K_0(D)$ has been shown to be isomorphic to $\mathbb{Z}\left[\frac{1}{p}\right]$ as an abelian group, the statements about ordered K-theory is proved as in the proof of Proposition 7.3, and the Cuntz semigroup is computed as in the proof of Corollary 7.4.

The computation of $K_0(D)$ is done in the same way as in the proof of Proposition 7.3 up to the point at which we found $K_0(C_0)$ and $K_0(C_0^{\gamma})$. Here we get

$$K_0(C_0) \cong \mathbb{Z}\left[\tfrac{1}{p}\right]^p \quad \text{and} \quad K_0(D) \cong \bigcap_{m=0}^{p-1} \ker(\mathrm{id} - K_0(\gamma^m)).$$

Also, we here find that $K_0(\gamma)\colon \mathbb{Z}\left[\frac{1}{p}\right]^p \to \mathbb{Z}\left[\frac{1}{p}\right]^p$ is given by

$$K_0(\gamma)(\eta_1, \eta_2, \ldots, \eta_p) = (\eta_p, \eta_1, \eta_2, \ldots, \eta_{p-1}).$$

Therefore $\mathrm{id} - K_0(\gamma)$ has the matrix

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The map $\eta \to (\eta, \eta, \ldots, \eta)$ is an isomorphism from $\mathbb{Z}\left[\frac{1}{p}\right]$ to $\ker(\mathrm{id} - K_0(\gamma))$, and one checks that its image is contained in $\ker(\mathrm{id} - K_0(\gamma^m))$ for all m. Therefore this map is an isomorphism from $\mathbb{Z}\left[\frac{1}{p}\right]$ to $\bigcap_{m=0}^{p-1} \ker(\mathrm{id} - K_0(\gamma^m))$. The rest of the computation of $K_0(D)$ follows the proof of Proposition 7.3.

It remains only to show that $D^{\mathrm{op}} \not\cong D$. As in the proof of Theorem 5.6, it is sufficient to prove a nonisomorphism of a von Neumann algebra with its opposite analogous to Proposition 5.4. We use the same notation as in Proposition 5.4 for the analogous maps in our situation. The analogs of the statements in Lemma 5.3 are that now

$$N = \left[\bigstar_1^p L^\infty([0,1])\right] \star \mathcal{L}(\mathbb{Z}_p),$$

and

$$\widetilde{\alpha}(\lambda_1(g)) = \lambda_2(g), \quad \widetilde{\alpha}(\lambda_2(g)) = \lambda_3(g), \quad \ldots, \quad \widetilde{\alpha}(\lambda_{p-1}(g)) = \lambda_p(g), \tag{8.1}$$

$$\widetilde{\alpha}(\lambda_p(g)) = \mathrm{Ad}(u_0)(\lambda_1(g)), \quad \text{and} \quad \widetilde{\alpha}(u_0) = e^{-2\pi i/p} u_0. \tag{8.2}$$

Moreover, $(\widetilde{\alpha})^p = \mathrm{Ad}(u_0)$.

Next, as in Theorem 5.6, the automorphism $\mathrm{Ad}(w) \circ \big(\widetilde{\alpha} \otimes \widetilde{\beta}\big)$ generates an action $\widetilde{\gamma}$ of $\mathbb{Z}_p$ on $N \overline{\otimes} R$, and $M = (N \overline{\otimes} R) \rtimes_{\widetilde{\gamma}} \mathbb{Z}_p$ is a factor of type II_1. The nonisomorphism $D^{\mathrm{op}} \not\cong D$ will follow from the nonisomorphism $M^{\mathrm{op}} \not\cong M$.

Our proof that $M^{\mathrm{op}} \not\cong M$ proceeds as follows. We give a recipe which starts with the factor M, just given as a factor of type II_1 with certain properties (see (1)–(4) below), and produces a subset $S_p(M)$ of $\mathbb{Z}_p$, which, when convenient, we identify with $\{0, 1, \ldots, p-1\}$. The important point is that this recipe does not depend on knowing any particular element, automorphisms, etc. of M. That is, if we start with some other factor of type II_1 which is isomorphic to M, then we get the same subset of $\{0, 1, \ldots, p-1\}$, regardless of the choice of isomorphism. When -1 is not a square mod p, we will show that the recipe also applies to M^{op} and gives a different subset, from which it will follow that $M^{\mathrm{op}} \not\cong M$.

We describe the construction first, postponing the proofs that the steps can be carried out and the result is independent of the choices made. Let P be a factor of type II_1 with separable predual and with the following properties:

(1) $\chi(P) \cong \mathbb{Z}_{p^2}$.
(2) The unique subgroup of $\chi(P)$ of order p is the image of a subgroup (not unique) of $\mathrm{Aut}(P)$ isomorphic to $\mathbb{Z}_p$.
(3) Let $\rho\colon \mathbb{Z}_p \to \mathrm{Aut}(P)$ come from a choice of the subgroup and isomorphism in (2). Form the crossed product $P \rtimes_\rho \mathbb{Z}_p$, and let $\widehat{\rho}\colon \widehat{\mathbb{Z}_p} \to \mathrm{Aut}(P \rtimes_\rho \mathbb{Z}_p)$ be the dual action. Then for every nontrivial element $l \in \widehat{\mathbb{Z}_p}$, the automorphism $\widehat{\rho}_l \in \mathrm{Aut}(P \rtimes_\rho \mathbb{Z}_p)$ can be factored as $\varphi \circ \psi$ for an approximately inner automorphism φ and a centrally trivial automorphism ψ.
(4) For any nontrivial element $l \in \widehat{\mathbb{Z}_p}$ and any factorization $\widehat{\rho}_l = \varphi \circ \psi$ as in (3), there is a unitary $z \in P \rtimes_\rho \mathbb{Z}_p$ such that $\psi^p = \mathrm{Ad}(z)$, and there is $k \in \{0, 1, \ldots, p-1\}$ (the obstruction to lifting, as used in [7], [8], and [35], except that Connes uses the number $e^{2\pi i k/p}$ instead of k) such that $\psi(z) = e^{2\pi i k/p} z$.

For P which satisfies (1)–(4), we take $S_p(P)$ to be the set of all values of $k \in \{0, 1, \ldots, p-1\}$ which appear in this way for any choice of the action $\rho\colon \mathbb{Z}_p \to \mathrm{Aut}(P)$, any nontrivial element $l \in \widehat{\mathbb{Z}_p}$, and any choice of the factorization $\widehat{\rho}_l = \varphi \circ \psi$ as in (3). We think of $S_p(P)$ as a subset of $\mathbb{Z}_p$ in the obvious way.

We check that if P satisfies (1) through (4), then so does P^{op}. For this purpose, we use the von Neumann algebra obtained from P by using the complex conjugate scalar multiplication and leaving all else unchanged. (We identify this algebra with P^{op} via the map $x \mapsto x^*$.) Scalar multiplication enters in the definition of $S_p(P)$ in only two places. The first is the definition of the dual action $\widehat{\rho}\colon \widehat{\mathbb{Z}_p} \to \mathrm{Aut}(P \rtimes_\rho \mathbb{Z}_p)$. However, the change is easily undone by applying the automorphism $l \mapsto -l$ of $\widehat{\mathbb{Z}_p}$. The other place is in the definition of the obstruction to lifting. So P^{op} satisfies (1) through (4), and we get, in the notation treating $S_p(\cdot)$ as a sunset of $\mathbb{Z}_p$,

$$S_p(P^{\mathrm{op}}) = \{-l \colon l \in S_p(P)\}. \tag{8.3}$$

In the rest of the proof, we show that our factor M above satisfies (1) through (4), and that moreover $S_p(M)$ can be computed using just one choice of $\rho\colon \mathbb{Z}_p \to \mathrm{Aut}(P)$ and, for each nontrivial element $l \in \widehat{\mathbb{Z}_p}$, one choice of the factorization $\widehat{\rho}_l = \varphi \circ \psi$ in (3) and one choice of the unitary z in (4). We then finish by computing $S_p(M)$.

Begin by observing that $N \cong \mathcal{L}(\mathcal{F}_p \star \mathbb{Z}_p)$ is full, that is, has no nontrivial central sequences. (See Lemma 3.2 and Remark 3.1 of [35].) It follows from the proof of Proposition 3.5 of [35] that $N \overline{\otimes} R$ has no nontrivial hypercentral sequences. The arguments of Section 5 of [35] can now be used to compute $\chi(M)$, and show that it is isomorphic to $\mathbb{Z}_{p^2}$, as required for (1). The arguments of Section 5 of [35] also show that the unique subgroup of order p is the image of the action $\sigma\colon \widehat{\mathbb{Z}_p} \to \mathrm{Aut}(M)$ obtained as the dual action on M coming from the isomorphism $M \cong (N \overline{\otimes} R) \rtimes_{\widetilde{\gamma}} \mathbb{Z}_p$. Thus, there exists at least one choice for ρ, namely σ composed with some isomorphism $\mathbb{Z}_p \to \widehat{\mathbb{Z}_p}$. This is (2).

We claim that the crossed product $M \rtimes_\rho \mathbb{Z}_p$ and the dual action $\widehat{\rho}\colon \widehat{\mathbb{Z}_p} \to \mathrm{Aut}(M \rtimes_\rho \mathbb{Z}_p)$ are uniquely determined up to conjugacy and automorphisms of $\mathbb{Z}_p$. This will show that $S_p(M)$ can be computed for any particular fixed choice of $\rho\colon \mathbb{Z}_p \to \mathrm{Aut}(M)$. There are two ambiguities in the choice of ρ. If we change the isomorphism of $\mathbb{Z}_p$ with the subgroup of $\chi(M)$ of order p, we are modifying ρ by an automorphism of $\mathbb{Z}_p$. The crossed product $M \rtimes_\rho \mathbb{Z}_p$ is the same, and the dual action is modified by the corresponding automorphism of $\widehat{\mathbb{Z}_p}$. Suppose, then, that we fix an isomorphism of $\mathbb{Z}_p$ with the subgroup of $\chi(M)$ of order p, but choose a different lift $\widetilde{\rho}$ to a homomorphism $\mathbb{Z}_p \to \mathrm{Aut}(M)$. Then Lemma 8.4 implies that the crossed products are isomorphic and the dual actions are conjugate. This proves the claim.

By Takesaki's duality theory (see Theorem 4.5 of [33]), there is an isomorphism

$$M \rtimes_\sigma \widehat{\mathbb{Z}_p} \cong (N \overline{\otimes} R) \otimes B(l^2(\mathbb{Z}_p)),$$

which identifies $g \mapsto \widehat{\sigma}_g$ with the tensor product $g \mapsto \widetilde{\gamma}_g \otimes \mathrm{Ad}(\lambda(g)^*)$ of $\widetilde{\gamma}$ and the conjugation by the left regular representation of $\mathbb{Z}_p$ on $l^2(\mathbb{Z}_p)$.

Now let $l \in \widehat{\mathbb{Z}_p}$. We claim that $\widehat{\sigma}_l$ can be written as a product $\varphi \circ \psi$ for an approximately inner automorphism φ and a centrally trivial automorphism ψ, and that this factorization is unique up to inner automorphisms. This will imply (3). We first consider uniqueness, which is equivalent to showing that every automorphism which is both approximately inner and centrally trivial is in fact inner. Since N is full, the decomposition in the proof of Lemma 3.6 of [35] can be used to show that every automorphism of $N \overline{\otimes} R$ which is both approximately inner and centrally trivial is in fact inner. Uniqueness now follows because $(N \overline{\otimes} R) \otimes B(l^2(\mathbb{Z}_p)) \cong N \overline{\otimes} R$.

For existence, since the approximately inner automorphisms form a normal subgroup of $\mathrm{Aut}\big(M \rtimes_\sigma \widehat{\mathbb{Z}_p}\big)$, it suffices to take l to be the standard generator of $\widehat{\mathbb{Z}_p}$. Equivalently, consider $\widetilde{\gamma} \otimes \mathrm{Ad}(\lambda(1)^*)$. We will take

$$\varphi = \big(\mathrm{Ad}(w) \circ \big(\mathrm{id}_N \otimes \widetilde{\beta}\big)\big) \otimes \mathrm{Ad}(\lambda(1)^*) \quad \text{and} \quad \psi = (\widetilde{\alpha} \otimes \mathrm{id}_R) \otimes \mathrm{id}_{B(l^2(\mathbb{Z}_p))}.$$

It is clear that $\widetilde{\gamma} \otimes \mathrm{Ad}(\lambda(1)^*) = \varphi \circ \psi$. The automorphism φ is approximately inner because, by construction, $\widetilde{\beta}$ is approximately inner. (In fact, by Theorem XIV.2.16 of [34], every automorphism of R is approximately inner.) To see that ψ is centrally trivial, we observe that, by the proof of Proposition 3.5 of [35], every central sequence in $N \overline{\otimes} R$ has the form $(1 \otimes x_n)_{n=1}^{\infty}$ for some central sequence $(x_n)_{n=1}^{\infty}$ in R. This proves the claim.

The obstruction to lifting for ψ (as in property (4)) is independent of the choice of z, by the discussion at the beginning of Section 1 of [9]. By Proposition 1.4 of [9] it is unchanged if ψ is replaced by $\mu \circ \psi \circ \mu^{-1}$ for any $\mu \in \mathrm{Aut}\big(M \rtimes_{\sigma} \widehat{\mathbb{Z}_p}\big)$, and also if ψ is replaced by $\mathrm{Ad}(y) \circ \psi$ for any unitary $y \in M \rtimes_{\sigma} \widehat{\mathbb{Z}_p}$. Since $\widehat{\sigma}$ is determined up to conjugacy and the centrally trivial factor of an automorphism is determined up to inner automorphisms, it follows that we can compute $S_p(M)$ by simply computing the obstructions to lifting for all powers ψ^l for a fixed choice of ψ and for $l = 1, 2, \ldots, p-1$. We can take

$$\psi = (\widetilde{\alpha} \otimes \mathrm{id}_R) \otimes \mathrm{id}_{B(l^2(\mathbb{Z}_p))},$$

for which $z = u_0 \otimes 1 \otimes 1$ has already been shown to be a unitary with $\psi^p = \mathrm{Ad}(z)$ and $\psi(z) = e^{-2\pi i/p} z$. Now one uses Equations (8.1) and (8.2) to check that

$$(\psi^l)^p = \mathrm{Ad}(z^l) \quad \text{and} \quad \psi^l(z^l) = e^{-2\pi i l^2/p} z^l.$$

Therefore (identifying $\{0, 1, \ldots, p-1\}$ with $\mathbb{Z}_p$ in the usual way)

$$S_p(M) = \big\{ -l^2 \colon l \in \mathbb{Z}_p \setminus \{0\} \big\}.$$

As observed in Equation (8.3) above, $S_p(M^{\mathrm{op}})$ is then given by

$$S_p(M^{\mathrm{op}}) = \big\{ l^2 \colon l \in \mathbb{Z}_p \setminus \{0\} \big\}.$$

Since we are assuming -1 is not a square mod p, we have $S_p(M^{\mathrm{op}}) \neq S_p(M)$, whence $M^{\mathrm{op}} \not\cong M$. □

9. Open problems

We discuss here some open questions on simple separable C*-algebras not isomorphic to their opposites. We think that with similar techniques to the ones used in the previous section for the C*-algebra D we can get other choices of K-theory as well. However, it seems that more complicated methods are required to solve the following problem.

Question 9.1. Let B be any UHF algebra. Is there a simple separable exact C*-algebra, not isomorphic to its opposite algebra, whose K-theory is the same as that of B, but which otherwise has all the properties of the algebra D constructed in this paper?

Of course, one can generalize this question, letting B be a simple unital AF algebra, or a simple unital AH algebra with no dimension growth and real rank zero. If one drops the requirement that D have real rank zero, there are even more choices for B.

New methods are needed to address the following two questions.

Question 9.2. Is there a simple separable purely infinite C*-algebra which is not isomorphic to its opposite algebra?

Question 9.3. Is there a simple separable nuclear C*-algebra which is not isomorphic to its opposite algebra?

We see no obvious obstruction to a positive answer to Question 9.2, especially since there are type III factors not isomorphic to their opposite algebras [8]. A positive answer to Question 9.3 would be much more surprising, in view of the Elliott program and the fact that all known invariants of simple nuclear C*-algebras, even the Cuntz semigroup, are unable to distinguish a C*-algebra from its opposite.

Acknowledgments

The second author would like to thank Prof. E. Kirchberg for useful discussions, and for suggestions helpful in proving Lemma 3.2.

Some of this work was carried out during the Fields Institute program on operator algebras of Fall 2007, and during a visit by the first author to the Fields Institute for the Spring 2008 Canadian Operator Symposium. Both authors are grateful to the Fields Institute for its hospitality.

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